Salem numbers and growth series of some hyperbolic graphs

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Abstract. Extending the analogous result of Cannon and Wagreich for the fundamental groups of surfaces, we show that, for the ℓ-regular graphs $\mathcal{X}_{\ell,m}$ associated to regular tessellations of the hyperbolic plane by m-gons, the denominators of the growth series (which are rational and were computed by Floyd and Plotnick (Floyd and Plotnick, 1994)) are reciprocal Salem polynomials. As a consequence, the growth rates of these graphs are Salem numbers. We also prove that these denominators are essentially irreducible (they have a factor of $X + 1$ when $m \equiv 2 \mod 4$; and when $\ell = 3$ and $m \equiv 4 \mod 12$, for instance, they have a factor of $X^2 - X + 1$). We then derive some regularity properties for the coefficients f_n of the growth series: they satisfy

$$K\lambda^n - R < f_n < K\lambda^n + R$$

for some constants $K, R > 0$, $\lambda > 1$.

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1. Introduction

We consider the graphs $\mathcal{X}_{\ell,m}$ ($\ell, m \geq 3$) defined by Floyd and Plotnick in (Floyd and Plotnick, 1994). These graphs are ℓ-regular and are the 1-skeletons of tessellations of the sphere (if $(\ell-2)(m-2) < 4$), the Euclidean plane (if $(\ell-2)(m-2) = 4$) or the hyperbolic plane (if $(\ell-2)(m-2) > 4$) by regular m-gons. These tessellations were studied by Coxeter (Coxeter, 1954). When $m = \ell = 4g$ with g at least two, $\mathcal{X}_{\ell,m}$ is the Cayley graph of the fundamental group $J_g = \pi_1(\Sigma_g)$ of an orientable closed surface Σ_g of genus g, with respect to the usual set of generators $S_g = \{a_1, b_1, \ldots, a_g, b_g\}$:

$$J_g = \left\langle a_1, b_1, \ldots, a_g, b_g \left| \prod_{i=1}^{g} [a_i, b_i] \right. \right\rangle.$$

The growth series for J_g with respect to S_g, namely

$$F_g(X) = \sum_{s \in J_g} X^{|s|} = \sum_{n=0}^{\infty} f_n X^n,$$

where $|s| = \min\{t : s = s_1 \ldots s_t, s_i \in S_g \cup S_g^{-1}\}$ denotes the word length of s with respect to S_g and $f_n = |\{s \in J_g : |s| = n\}|$, was computed by Cannon and Wagreich in (Cannon, 1983) and (Cannon and Wagreich, 1992) and shown to be rational, indeed

$$F_g(X) = \frac{1 + 2X + \cdots + 2X^{2g-1} + X^{2g}}{1 - (4g-2)X - \cdots - (4g-2)X^{2g-1} + X^{2g}};$$

moreover they showed that the denominator is a Salem polynomial. It was later shown by Floyd (Floyd, 1992) and Parry (Parry, 1993) that the denominators of the growth series of Coxeter groups are also Salem polynomials.

In (Floyd and Plotnick, 1987) and (Floyd and Plotnick, 1994), Floyd and Plotnick, among other things, extended the calculations of Cannon and Wagreich to the family $\mathcal{X}_{\ell,m}$. Fixing arbitrarily a base point $* \in V(\mathcal{X}_{\ell,m})$ and denoting by $|x|$ the graph distance between the vertices x and $*$, they obtained the following formulæ for the growth series $F_{\ell,m}(X) = \sum_{x \in V(\mathcal{X}_{\ell,m})} X^{|x|}$; for m even, say $m = 2w$:

$$F_{\ell,m}(X) = \frac{1 + 2X + \cdots + 2X^{w-1} + X^w}{1 - (\ell-2)X - \cdots - (\ell-2)X^{w-1} + X^w} \tag{1}$$

and, for m odd, say $m = 2w + 1$:

$$F_{\ell,m}(X) = \frac{1 + 2X + \cdots + 2X^{w-1} + 4X^w + 2X^{w+1} + \cdots + 2X^{2w-1} + X^{2w}}{1 - (\ell-2)X - \cdots - (\ell-4)X^w - \cdots - (\ell-2)X^{2w-1} + X^{2w}}. \tag{2}$$

Our main result is the following:

THEOREM 1. *The denominators of the growth series $F_{\ell,m}$ are reciprocal Salem polynomials. After simplification by $X + 1$ in case $m \equiv 2 \mod 4$, they are irreducible, except for the following exceptional cases:*

1. *$\ell = 3$ and $m \equiv 4 \mod 12$; there is a factor of $X^2 - X + 1$;*

2. *$\ell = 3$ and $m \equiv 3 \mod 6$; there is a factor of $X^2 + X + 1$;*

3. *$\ell = 3$ and $m \equiv 5 \mod 20$; there is a factor of $X^4 - X^3 + X^2 - X + 1$;*

4. *$\ell = 4$ and $m \equiv 3 \mod 8$; there is a factor of $X^2 + 1$;*

5. *$\ell = 5$ and $m \equiv 3 \mod 12$; there is a factor of $X^2 - X + 1$.*

This theorem extends the results in Section 4 of (Floyd and Plotnick, 1994) which show, for even m, that the denominator of $F_{\ell,m}$ is a product of an irreducible Salem polynomial and distinct cyclotomic polynomials.

COROLLARY 2. *The growth rates of the graphs $\mathcal{X}_{\ell,m}$ are Salem numbers.*

We thus obtain more precise information about the growth coefficients:

COROLLARY 3. *If $F_{\ell,m}(X) = \sum_{n \geq 0} f_n X^n$, then there exist constants $K > 0$, $\lambda > 1$ and $R > 0$ such that*

$$K\lambda^n - R < f_n < K\lambda^n + R$$

holds for all n. Moreover λ is a Salem number.

This improves, for the graphs $\mathcal{X}_{\ell,m}$, on a general result by Coornaert (Coornaert, 1993) for non-elementary hyperbolic groups, which asserts that there exist constants $\lambda > 1$ and $0 < K_1 < K_2$ such that

$$K_1\lambda^n < f_n < K_2\lambda^n$$

for all n.

Note that Corollary 3 does not hold for a general presentation of a hyperbolic group, nor even of a virtually free group. Consider for instance the modular group $PSL_2(\mathbb{Z}) = \langle a, b \,|\, a^2 = b^3 = 1 \rangle$. The growth coefficients f_n satisfy

$$K\phi^n - 2 < f_n < K\phi^n + 2 \qquad \text{with } K = \frac{3 + \sqrt{5}}{\sqrt{5}}, \phi = \frac{1 + \sqrt{5}}{2}$$

for the generating set $\{a, ab, (ab)^{-1}\}$ (note that ϕ is a Salem number), but for the generating set $\{a, b, b^{-1}\}$ are

$$f_n = \begin{cases} 2 \cdot \sqrt{2}^n & \text{if } n \text{ is even,} \\ 3/\sqrt{2} \cdot \sqrt{2}^n & \text{if } n \text{ is odd.} \end{cases}$$

These computations are due to Machì; see (Harpe, 2000, VI.7). It follows that Corollary 3 does not extend to arbitrary hyperbolic group presentations (indeed, $\sqrt{2}$ is not even a Salem number).

2. Salem Polynomials

We recall a few facts on Salem polynomials; one might also consult (Bertin et al., 1992, § 5.2) or the original paper (Salem, 1945). A polynomial $f(X) = f_0 + f_1 X + \cdots + f_n X^n$ ($f_n \neq 0$) is *reciprocal* if $f_i = f_{n-i}$ for all $i = 0, 1, \ldots, n$; equivalently if $X^n f(X^{-1}) = f(X)$.

A polynomial $f(X) \in \mathbb{Z}[X]$ is a *Salem polynomial* if it is monic, admits exactly one root λ of modulus $|\lambda| > 1$, and this root is simple; this λ is then necessarily real. If moreover f is reciprocal, then $1/\lambda$ is also a root of f and these are the only roots of f not on the unit circle.

If f is a reciprocal polynomial of odd degree, then $f(-1) = 0$, so $X + 1$ divides f. There is therefore no real limitation in considering reciprocal Salem polynomials of even degree.

If f is a Salem polynomial and g is a cyclotomic polynomial, then fg is again a Salem polynomial, and reciprocally any Salem polynomial can be factored as a product of an irreducible Salem polynomial and cyclotomic polynomials.

A real number λ is called a *Salem number* if $\lambda > 1$, λ is an algebraic integer, and all its conjugates except $\lambda^{\pm 1}$ have absolute value 1; equivalently if $\lambda > 1$ is the root of a Salem polynomial.

For instance, the reciprocal Salem polynomials of degree 2 are the $X^2 - aX + 1$ for all $a \in \mathbb{Z}$ with $a \geq 3$. The corresponding Salem numbers are the $(a + \sqrt{a^2 - 4})/2$. The Salem polynomials of degree 2 are all the $X^2 - aX + b$ subject to $a > |1 + b|$ and $b \neq 0$.

Denoting by

$$\Phi(z) = \frac{z - i}{z + i}$$

the Cayley transform (Rudin, 1991, 13.17) mapping the extended real axis $\mathbb{R} \cup \{\infty\}$ onto the unit circle $\mathbb{T}$, we give the following

DEFINITION 4. *Let f be a polynomial of degree n. Its* Cayley transform *is the polynomial*

$$C(f)(X) = (X + i)^n f(\Phi(X)).$$

Note that if f is real and reciprocal, its Cayley transform will again be real.

The proof of the following characterization of reciprocal Salem polynomials is straight-forward:

THEOREM 5. *Let f be a monic integral reciprocal polynomial of degree n. Then f is a Salem polynomial if and only if the polynomial $C(f)$ has exactly $n - 2$ real roots (its last two roots are then complex conjugate).*

3. The Denominators of the Growth Series $F_{\ell,m}$

The objective of this section is to prove that the denominator of $F_{\ell,m}$ is (after a possible division by $X + 1$ depending on the parity of its degree m) a reciprocal Salem polynomial. For this purpose define the following auxiliary polynomials, for $a \in \mathbb{Z}$ and $b, k \in \mathbb{N}$:

$$p_b(X) = 1 + 2X + 2X^2 + \cdots + 2X^{b-1} + X^b = (1 - X^b)\frac{1 + X}{1 - X}, \tag{3}$$

$$q_b(X) = 1 + X^b, \tag{4}$$

$$r_{a,b}(X) = 1 + aX + aX^2 + \cdots + aX^{b-1} + X^b, \tag{5}$$

$$r_{a,b;\,k}(X) = 1 + aX + aX^2 + \cdots + aX^{b-1} + X^b + kX^{b/2} \quad (b \text{ even}). \tag{6}$$

LEMMA 1. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions, such that the following holds: f has t real zeroes $\rho_1 < \cdots < \rho_t$, and g has a transverse zero in each interval $]\rho_i, \rho_{i+1}[$ and no other zero in $[\rho_1, \rho_t]$. Then for any $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ the function $\alpha f + \beta g$ has at least $t - 1$ real zeroes.*

Proof. The function $\alpha f + \beta g$ has opposite signs at ρ_i and ρ_{i+1}, so has a zero in $]\rho_i, \rho_{i+1}[$.

LEMMA 2. *If $a \in \mathbb{Z}$, $b \in 2\mathbb{N}$, and $2 + a(b-1) < 0$, then $r_{a,b}$ is a reciprocal Salem polynomial.*

Proof. First write

$$r_{a,b}(X) = \frac{a}{2} p_b(X) + (1 - \frac{a}{2}) q_b(X).$$

As p_b and q_b are symmetric polynomials, their Cayley transforms are polynomials with real coefficients. The zeroes of p_b are -1 and the $e^{2i\pi \frac{k}{b}}, k \in \{1, \ldots, b-1\}$; the zeroes of q_b are the $e^{i\pi \frac{2k+1}{b}}, k \in \{0, \ldots, b-1\}$. As these zeroes are interleaved on the unit circle, the zeroes of their Cayley transforms will be interleaved on the real axis, and we may apply Lemma 1 to conclude that $r_{a,b}(X)$ has at least $b-1$ zeroes on the unit circle. It has precisely $b-1$ zeroes there because $r_{a,b}(1) = 2 + a(b-1) < 0$ and $\lim_{x\to\infty} r_{a,b}(x) = +\infty$, so that $r_{a,b}$ has a zero in $]1, \infty[$.

LEMMA 3. *Let $f(X) \in \mathbb{Z}[X]$ be a reciprocal Salem polynomial. Let $g(X) \in \mathbb{Z}[X]$ be a polynomial of degree less than $\deg(f)$, such that $f + g$ is reciprocal. Consider for $\epsilon \in \mathbb{R}$ the perturbation $f_\epsilon = f + \epsilon g \in \mathbb{R}[X]$. Let $k \in \mathbb{N}$ be such that f_ϵ has only simple roots for all $\epsilon \in [0, k]$. Then f_k is a reciprocal Salem polynomial.*

Proof. Let F and F_ϵ be the Cayley transforms of f and f_ϵ. Then F has real roots except two which are complex conjugate, and has real coefficients. By assumption, the discriminant of F_ϵ does not change its sign on $[0, k]$ and thus F_k has real roots except two which are still complex conjugate. Taking the inverse Cayley transform yields f_k which has all its roots on the unit circle except two, and thus is a Salem polynomial.

LEMMA 4. *If $a \in \mathbb{Z}$, $b \in 2\mathbb{N}$ with $2 + a(b-1) < 0$, and $k \in \mathbb{N}$ with $k \leq \min\{2 - a, -2 - a(b-1)\}$, then $r_{a,b;\,k}$ is a reciprocal Salem polynomial.*

Proof. Denote $\min\{2 - a, -2 - a(b-1)\}$ by K. In view of Lemma 3, it suffices to prove that $r_{a,b;\,\epsilon} = r_{a,b} + \epsilon X^{b/2}$ is simple for all $\epsilon \in [0, K[$. For this purpose consider the function $f(X) = X^{-\frac{b}{2}} r_{a,b}(X) = X^{-\frac{b}{2}} \left(1 + X^b + a\frac{1-X^b}{1-X}\right)$, considered as a function on the circle ($f : \mathbb{T} \to \mathbb{C}$).

Since f is real and $f(X^{-1}) = f(X)$, it satisfies $f(\mathbb{T}) \subset \mathbb{R}$: for any $\xi \in \mathbb{T}$,

$$\overline{f(\xi)} = \overline{f}(\overline{\xi}) = f(\xi^{-1}) = f(\xi).$$

We show that $r_{a,b;\,\epsilon}$ is simple by showing that $f + \epsilon$ has only simple zeroes on $\mathbb{T}$, or equivalently that f attains values greater than ϵ between its zeroes on $\mathbb{T}$.

Since $r_{a,b}$ is a reciprocal Salem polynomial, it has $b-2$ zeroes on $\mathbb{T}$, so f has also $b-2$ zeroes on $\mathbb{T}$. Consider the points $\xi_j = e^{2i\pi j/b} \in \mathbb{T}$, for $j \in \{1, \ldots, b-1\}$. We have

$$f(\xi_j) = \xi_j^{-\frac{b}{2}} \left(\xi_j^b + 1 + a\frac{\xi_j^b - \xi_j}{\xi_j - 1}\right) = (-1)^j (2 - a),$$

so the zeroes of f are separated by extrema of at least $\pm(2 - a)$.

Finally, since $f(1) + \epsilon \leq \epsilon - K < 0$, the two real zeroes of $f + \epsilon$ remain always separated by the unit circle. The zeroes of $f + \epsilon$ (and thus of $r_{a,b;\,\epsilon}$) on $\mathbb{T}$ therefore remain simple.

We now turn to the factorization of the $r_{a,b}$. Let Q be any reciprocal Salem polynomial, let λ be the Salem number associated to Q, and let S be the minimal polynomial of λ. Then we have a factorization $Q = ST$, where T, having only roots on the unit circle, is a product of cyclotomic polynomials, in virtue of the theorem of Kronecker (Kronecker, 1899, Vol. III, Part I, pages 47–110). We show that this show that this factor T is either 1 or $X + 1$, depending on the parity of b:

PROPOSITION 6. *Suppose $|a - 1| \geq 2$. Then the only cyclotomic polynomials dividing the $r_{a,b}$ defined in (5) are $X + 1$ when b is odd, and $X^2 - X + 1$ when $a = -1$ and $b \equiv 2$ mod 6, with the exception $a = -2, b = 2$ when $r_{a,b} = (X - 1)^2$ is not a Salem polynomial.*

Proof. Set $f(X) = r_{a,b}(X)$. Clearly $f(-1) = 0$ precisely when b is odd, and $f(1) \neq 0$. Suppose now that η is a root of unity of order $n > 2$ satisfying $f(\eta) = 0$. We may suppose, by direct computation, that n does not divide $b - 1$. Let ξ be an algebraic conjugate of η satisfying $|\xi^{b-1} - 1| \geq 1$. Since

$$\left| \frac{\xi^{b+1} - 1}{\xi^{b-1} - 1} \right| = \left| \xi \left(1 - a + \frac{f(\xi)}{\xi + \cdots + \xi^{b-1}} \right) \right| = |a - 1|,$$

and $|\xi^{b+1} - 1| \leq 2$, we have $|a - 1| \leq 2$. Now in case $|a - 1| = 2$ and ξ is such a root, we must have $|\xi^{b+1} - 1| = 2$ whence $\xi^{b+1} = -1$. Then we have $\xi^b - \xi = (1 - a)/2 = \pm 1$, or, after multiplication by ξ and simplification, $\xi^2 \pm \xi + 1 = 0$, so ξ is of degree 3 or 6. We check by substitution in f that indeed $\xi^2 - \xi + 1$ divides f when $b \equiv 2 \mod 6$. We have shown that the only roots of unity of degree greater than 2 are the sixth, and that they occur only in very special cases.

A similar, but more complicated, result holds for $r_{a,b;\,2}$; then there are special cases for $-3 \leq a \leq -1$, with factors T depending on the value of b modulo 4, 6 and 10, as mentioned in the statement of the theorem; we omit the uninteresting details and quote the result without proof.

Proof. [Proof of Theorem 1] If m is even the denominator of $F_{\ell,m}$ is $r_{2-\ell,\frac{m}{2}}$, by (1), and thus is a (reciprocal) Salem polynomial by Lemma 2. If m is odd, this denominator is $r_{2-\ell,m-1;\,2}$, by (2), and thus is a (reciprocal) Salem polynomial, by Lemma 4. It is irreducible, by Proposition 6.

4. The Growth of the Graphs $\mathcal{X}_{\ell,m}$

Recall the following well-known fact from (Graham et al., 1994, p. 341):

LEMMA 5. *Let $f(X) = P(X)/Q(X) = \sum_{n \geq 0} f_n X^n$ be a rational function of X, where P and Q are complex polynomials and $Q(X) = \prod_{i=1}^{r} (X - \alpha_i)^{\nu_i}$ with distinct $\alpha_i \in \mathbb{C}$, namely $\alpha_i \neq \alpha_j$ when $i \neq j$. Then there exist polynomials $R_1, \ldots, R_r \in \mathbb{C}[X]$ such that $f_n = \sum_{i=1}^{r} R_i(n)/\alpha_i^n$. Moreover the degree of R_i is strictly smaller than ν_i, for all i. In particular, if all poles of f are simple, then the R_i are constant.*

From this, we derive the following result:

THEOREM 7. *Let*

$$f(X) = \frac{P(X)}{Q(X)} = \sum_{n \geq 0} f_n X^n$$

be a rational function of X, where $P \in \mathbb{Z}[X]$ and Q is a Salem polynomial. Then there exist a constant $K > 0$ and a polynomial R such that for all n we have

$$K\lambda^n - R(n) < f_n < K\lambda^n + R(n),$$

where $\lambda > 1$ is the Salem number associated to Q. The degree of R is strictly less than the maximal multiplicity of f's poles. Thus if moreover all poles of f are simple, then there exist constants $\lambda > 1$, K and R such that

$$K\lambda^n - R < f_n < K\lambda^n + R.$$

Proof. Apply Lemma 5 to f to obtain polynomials $R_1, \ldots, R_r$. Without loss of generality we may assume that α_r is the only pole of f inside the unit circle. Thus $K := R_r$ is a constant. Set also $\lambda = 1/\alpha_r$. Writing $R_i(n) = \sum b_{ij} n^j$, we define polynomials S_i by $S_i(n) = \sum |b_{ij}| n^j$, and we let

$$R(n) := \sum_{i=1}^{r-1} S_i(n).$$

Then $|R_i(n)/\alpha_i^n| \leq S_i(n)$, and

$$|f_n - K\lambda^n| = |f_n - \frac{R_r(n)}{\alpha_r^n}| \leq R(n).$$

If all poles of f are simple, then the R_i are constants and so is R.

Corollary 3 follows from the previous theorem.

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